Year: 2003

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Originally published at:
Kappeler, T; Topalov, P (2003). Riccati Representation for Elements in H-1(T) and its Applications. Pliska Bulgarski Matematicheski Studii, 15:171-188.

Riccati representation for elements in $H^{-1}(\mathbb{T}^1)$ and its applications

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September 3, 2002

Abstract

The paper is concerned with the spectral properties of the Schrödinger operator $L_q \stackrel{\text{def}}{=} -\frac{d^2}{dx^2} + q$ with periodic potential q from the Sobolev space $H^{-1}(\mathbb{T}^1)$. We obtain asymptotic formulas and a priori estimates for the periodic and Dirichlet eigenvalues which generalize known results for the case of potentials $q \in L_0^2(\mathbb{T}^1)$. The key idea is to reduce the problem to a known one – the spectrum of the impedance operator – via a nonlinear analytic isomorphism of the Sobolev spaces $H_0^{-1}(\mathbb{T}^1)$ and $L_0^2(\mathbb{T}^1)$.

1 Introduction

The present paper is devoted to the spectral properties of the Schrödinger operator $L_q \stackrel{\text{def}}{=} -\frac{d^2}{dx^2} + q$ with "singular" potential q from the Sobolev space $H^{-1}(\mathbb{T}^1)$. We prove the following results:

- periodic and Dirichlet spectrum of L_q are bounded from below and discrete;

[*]supported in part by European Research Training Network HPRN-CT-1999-00118

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- asymptotic formulas for the periodic and Dirichlet eigenvalues of L_q;
- asymptotic formulas for the gap lengths;
- a priori estimates of the potential q in terms of the gap lengths.

Our approach is based on a nonlinear representation - clearly of independent interest - of the elements of the Sobolev space $H^{-1}(\mathbb{T}^1)$ in terms of a unique function from $L^2(\mathbb{T}^1)$, refered to as Riccati representation.

To state our results precisely let us first introduce some notations used throughout the paper. Denote by $h^{\beta,n}$ ($\beta \in \mathbb{R}, n \in \mathbb{Z}$) the Hilbert space of sequences $\{x_k\}_{k\in\mathbb{Z}} \subset \mathbb{C}$ with finite norm $||x||_{\beta,n} \stackrel{\text{def}}{=} (\sum_{k\in\mathbb{Z}} \langle k+n\rangle^{2\beta} |x_k|^2)^{1/2}$, where $\langle s\rangle \stackrel{\text{def}}{=} |s|+1$. The scalar product in $h^{\beta,n}$ is given by $(x,y)_{\beta,n} \stackrel{\text{def}}{=} \sum_{k\in\mathbb{Z}} \langle k+n\rangle^{2\beta} x_k \bar{y}_k$. By definition, $h^\beta \stackrel{\text{def}}{=} h^{\beta,0}$.

By $\mathbb{T}^1_l$, $l > 0$, we denote the one-dimensional torus $\mathbb{T}^1_l \stackrel{\text{def}}{=} \mathbb{R}/l\mathbb{Z}$. The Sobolev spaces $H^m(\mathbb{T}^1_l)$ and $H^m[0,1]$, $m \in \mathbb{N}$, are defined by $H^m(\Omega) \stackrel{\text{def}}{=} \{f : \Omega \to \mathbb{R} \mid f^{(k)} \in L^2(\Omega),\ k = 0,1,...,m\}$ where $f^{(k)}$ is the k'th distributional derivative of the function f and Ω denotes the torus $\mathbb{T}^1_l$ or the interval $[0,1]$ respectively. The scalar product in $H^m(\Omega)$ is defined by $(f,g)_m \stackrel{\text{def}}{=} \sum_{k=0}^m (f^{(k)}, g^{(k)})$ where $(\cdot,\cdot)$ denotes the L^2-scalar product. For real $\alpha \geq 0$ the Sobolev spaces $H^\alpha(\mathbb{T}^1_l)$ and $H^\alpha[0,1]$ can be defined in a standard manner, for example, by interpolation (see [10], Chapter 1). By definition, $H^{-\alpha}(\Omega)$ is the dual space of $H^\alpha_c(\Omega)$, i.e. $H^{-\alpha}(\Omega) \stackrel{\text{def}}{=} (H^\alpha_c(\Omega))'$ where $H^\alpha_c(\Omega)$ is the closure in $H^\alpha(\Omega)$ of the space $C^\infty_c(\Omega)$ of smooth functions with compact support in the interior of Ω. The norm of f in $H^\alpha(\Omega)$ is denoted by $||f||_\alpha$ and for $\alpha = 0$ we write $||f|| = ||f||_0$. The distributional derivative $\frac{d}{dx} : H^m(\Omega) \to H^{m-1}(\Omega)$, $m \in \mathbb{N}$, can be extended in a natural way to a bounded operator $\frac{d}{dx} : H^\alpha(\Omega) \to H^{\alpha-1}(\Omega)$ for arbitrary $\alpha \in \mathbb{R}$. By $H^\alpha_0(\mathbb{T}^1_l)$ we denote the linear manifold of elements $f \in H^\alpha(\mathbb{T}^1_l)$ with mean value zero, $[f] \stackrel{\text{def}}{=} \int_{\mathbb{T}^1_l} f dx = 0$. Note that the Sobolev spaces $H^\alpha(\mathbb{T}^1_l)$, $\alpha \in \mathbb{R}$, can be identified (up to an equivalence of the norms) with the space of Fourier series $\sum_{k\in\mathbb{Z}} \hat{f}_k e^{2i\pi x/l}$ whose Fourier coefficients $\{\hat{f}_k\}_{k\in\mathbb{Z}}$ have finite h^α-norm.

Let $\rho \in H^1(\mathbb{T}^1)$ with $\min_{x\in\mathbb{T}^1} \rho(x) > 0$ and define $r \stackrel{\text{def}}{=} \rho'/\rho$. By the transformation $y \stackrel{\text{def}}{=} \rho\tilde{y}$ the equation $-\tilde{y}'' - 2r\tilde{y}' = \lambda\tilde{y}$ becomes the Schrödinger equation $-y'' + qy = \lambda y$ with $q = r' + r^2$. It was conjectured in [3] that the spectral properties of the Schrödinger operator L_q could be deduced from the spectral properties of the impedance operator $T_r(u) \stackrel{\text{def}}{=} -(\rho^2 u')'/\rho^2$, whose spectral theory is well-developed ([9, 1, 2, 6]). In the present paper we prove that any $q \in H^{\alpha-1}(\mathbb{T}^1)$, $\alpha \geq 0$, can be represented in a unique way in the form $q = r' + r^2 + const$, $r \in H_0^\alpha(\mathbb{T}^1)$. The uniqueness means that the function $r \in H_0^\alpha(\mathbb{T}^1)$ and the constant *const* are uniquely determined by q. More precisely, we prove that the *Riccati map* $r \mapsto r' + r^2 - ||r||^2$ maps $H_0^\alpha(\mathbb{T}^1)$ onto $H_0^{\alpha-1}(\mathbb{T}^1)$ where $||r||^2 = \int_0^1 r^2(x)dx$. In fact it is a real-analytic isomorphism between $H_0^\alpha(\mathbb{T}^1)$ and $H_0^{\alpha-1}(\mathbb{T}^1)$ (see Section 3). Our proof is elementary and based on the spectral properties of the Schrödinger operator. In particular, the properties of the first eigenvalue $\lambda_0(q)$ of the Schrödinger operator and the corresponding normalized eigenfunction $f_0(q)$ are essential for our approach.

The main result of this paper is summarized in the following theorem.

Theorem 1. *The Riccati map $R : L_0^2(\mathbb{T}^1) \to H_0^{-1}(\mathbb{T}^1)$ is a real analytic isomorphism. For any $r \in L_0^2(\mathbb{T}^1)$, the impedance operator T_r and the associated Schrödinger operator L_q with $q = R(r)$ have, up to a translation by $||r||^2$, the same periodic spectrum. Moreover, $R^{-1}(q)$ is given by*

$$R^{-1}(q) = f_0'(\cdot, q)/f_0(\cdot, q)$$

where $f_0(\cdot, q)$ is an eigenfunction corresponding to the first (lowest) periodic eigenvalue $\lambda_0(q)$ of L_q which can be proved to vanish nowhere.

The paper is organized as follows. In Section 2 we prove auxiliary results needed for the proof of Theorem 1 in Section 3. Section 4 is devoted to applications of the Riccati representation. In Section 4.1 we prove spectral results for the Hill operator L_q. The Dirichlet spectrum of L_q is considered in Section 4.2. The results there generalize results of [13] to the case of singular potentials $q \in H^{-1}(\mathbb{T}^1)$. Finally, we extend the notion of discriminant to potentials in $H^{-1}(\mathbb{T}^1)$ and prove the isospectral invariance of the Riccati map (Section 4.4). To make the paper self-contained we include Appendices A, B and C where we present results on the spectrum of T_r and L_q used in this paper.

The spectral theory of operators L_q with singular potentials $q \in H^\alpha(\mathbb{T}^1)$, $\alpha > -1$, was developed in [3] and [4] and was partly motivated by the construction of action-angle coordinates on the phase space $H^\alpha(\mathbb{T}^1)$ for the Korteweg-de Vries equation (KdV) – see the introduction in [3]. In a subsequent paper we want to extend the construction of action-angle coordinates for potentials in $H^{-1}(\mathbb{T}^1)$. In the stage of finishing this paper we were informed by E. Korotyaev that by different proofs, he showed that the Riccati map is a real analytic isomorphism and obtained similar results for the periodic spectrum of L_q ([8]).

2 L_q associated to T_r

For any given $r \in L^2_0(\mathbb{T}^1)$ denote by T_r the impedance operator

$$T_r(u) \stackrel{\text{def}}{=} -\frac{1}{\rho^2}(\rho^2 u')' = -u'' - 2ru' \tag{1}$$

on $L^2(\mathbb{T}^1_2)$ with domain $\texttt{Dom}(T_r) = H^2(\mathbb{T}^1_2)$. Here ρ is the absolutely continuous, 1-periodic, positive function given by $\rho(x) \stackrel{\text{def}}{=} \exp\left(\int\limits_0^x r(v)dv\right)$. In particular, $\rho \in H^1(\mathbb{T}^1)$ and $\rho' = r\rho$. Note that T_r is an operator with compact resolvent, non-negative, and symmetric with respect to the inner product $(f,g)_\rho \stackrel{\text{def}}{=} \int_0^2 fg\rho^2 dx$ on $L^2(\mathbb{T}^1_2)$. Hence the spectrum $\texttt{spec}(T_r)$ is discrete, real and non-negative. It turns out to be of the form $\texttt{spec}(T_r) = \{0 = \tilde{\lambda}_0(r) < \tilde{\lambda}_1(r) \leq \tilde{\lambda}_2(r) \leq ...\}$. The corresponding eigenspaces are of finite dimension, and $\tilde{\lambda}_k(r) \to \infty$ as $k \to \infty$ (see Appendix A).

For any $q \in H^{-1}(\mathbb{T}^1)$ we denote by L_q the Hill operator

$$L_q \stackrel{\text{def}}{=} -\frac{d^2}{dx^2} + q \tag{2}$$

viewed as an operator on the space $H^{-1}(\mathbb{T}^1_2)$ with domain $\texttt{Dom}(L_q) = H^1(\mathbb{T}^1_2)$. The classical spectral theory of Hill's operator can be extended for such singular potentials (see Appendix B). It is proved in Appendix B, Lemma 8, that the spectrum of L_q is discrete, real, and of the form $\texttt{spec}(L_q) = \{\lambda_0(q) \leq \lambda_1(q) \leq \lambda_2(q) \leq ...\}$, the corresponding eigenspaces are of finite dimension, and $\lambda_k(q) \to \infty$ as $k \to \infty$.

Clearly, for any $r \in L^2_0(\mathbb{T}^1)$, $r^2 - ||r||^2$ defines a bounded linear functional on $H^1(\mathbb{T}^1)$ satisfying $\langle r^2 - ||r||^2, 1\rangle = 0$ where $||r||^2 \stackrel{\text{def}}{=} \int_0^1 r^2(x)dx$. Hence, $r^2 - ||r||^2$ is an element in $H^{-1}_0(\mathbb{T}^1)$ and one can introduce the nonlinear map

$$R : L^2_0(\mathbb{T}^1) \to H^{-1}_0(\mathbb{T}^1),\ r \mapsto r' + r^2 - ||r||^2, \tag{3}$$

refered to as the Riccati map. The following result shows how the operators T_r and L_q are related if $q = R(r)$.

Lemma 1. *Let $r \in L^2_0(\mathbb{T}^1)$ and assume that $q \in H^{-1}_0(\mathbb{T}^1)$ satisfies Riccati's equation $q = R(r)$. Then*

(a) $\mathtt{spec}(T_r) = ||r||^2 + \mathtt{spec}(L_q)$;

(b) for any $k \geq 0$, the eigenspaces $V_{\lambda_k}(L_q)$ and $V_{\lambda_k + ||r||^2}(T_r)$ have the same dimension;

(c) the $L^2(\mathbb{T}^1)$-norm of r coincides with the absolute value of the first eigenvalue $\lambda_0(q)$ of Hill's operator L_q, i.e. $||r||^2 = |\lambda_0(q)| = -\lambda_0(q)$;

(d) the first eigenvalue $\lambda_0(q)$ of the operator L_q is simple and $f_0 \stackrel{\text{def}}{=} \rho/||\rho||$ is the corresponding eigenfunction normalized by $||f_0||^2 = 1$ and $f_0(0) > 0$. Hence, f_0 is in $H^1(\mathbb{T}^1)$, doesn't vanish on $\mathbb{T}^1$ and satisfies $r = \frac{f_0'}{f_0}$.

Proof. Let $f_k \in H^1(\mathbb{T}^1_2)$ be an eigenfunction of L_q with eigenvalue λ_k. Consider the function $\tilde{f}_k \stackrel{\text{def}}{=} \Phi_\rho(f_k) \stackrel{\text{def}}{=} f_k/\rho \in H^1(\mathbb{T}^1_2)$. Note that T_r extends in a natural way to a continuous operator $T_r : H^1(\mathbb{T}^1_2) \to H^{-1}(\mathbb{T}^1_2)$.

By Lemma 2 below, Riccati's map takes the form $(\rho''/\rho) - ||r||^2 = q$ where $\rho(x) \stackrel{\text{def}}{=} \exp(\int_0^x r(v)dv)$. In this way we obtain

$$\begin{aligned}
T_r(\tilde{f}_k) &\stackrel{\text{def}}{=} -\frac{1}{\rho^2}(\rho^2 \frac{f_k'}{\rho} - f_k \rho')' \\
&= -\frac{1}{\rho^2}(\rho f_k'' - f_k \rho'') \\
&= \frac{1}{\rho}(-f_k'' + \frac{\rho''}{\rho} f_k) \\
&= (\lambda_k + ||r||^2)\tilde{f}_k. \qquad (4)
\end{aligned}$$

It follows from the above formulas that $T_r(\tilde{f}_k) \stackrel{\text{def}}{=} -(\rho^2 \tilde{f}_k')'/\rho^2 \in H^1(\mathbb{T}^1_2)$. Using that ρ^2 is in $H^1(\mathbb{T}^1_2)$ and $H^1(\mathbb{T}^1_2)$ is an algebra, we obtain that $\rho^2 \tilde{f}_k' \in H^2(\mathbb{T}^1_2)$ and hence $\tilde{f}_k' \in H^1(\mathbb{T}^1_2)$. Therefore, we have proved that $\tilde{f}_k$ is in $H^2(\mathbb{T}^1_2)$ and thus an eigenfunction of T_r with eigenvalue $\tilde{\lambda}_k \stackrel{\text{def}}{=} \lambda_k + ||r||^2$.

Conversely, let $\tilde{f}_k \in H^2(\mathbb{T}^1_2)$ be an eigenfunction of the impedance operator T_r with eigenvalue $\tilde{\lambda}_k$. Then $f_k \stackrel{\text{def}}{=} \Phi_\rho^{-1}(\tilde{f}_k) \stackrel{\text{def}}{=} \rho \tilde{f}_k$ is in $H^1(\mathbb{T}^1_2)$. Using Riccati's equation $(\rho''/\rho) - ||r||^2 = q$ and Lemma 2 below, we obtain

$$L_q(f_k) = -\rho'' \tilde{f}_k - 2\rho' \tilde{f}_k' - \rho \tilde{f}_k'' + q\rho \tilde{f}_k$$

$$
\begin{aligned}
&= \rho((-\tilde{f}_k'' - 2\frac{\rho'}{\rho}\tilde{f}_k') - ||r||^2\tilde{f}_k) \\
&= (\tilde{\lambda}_k(r) - ||r||^2)f_k.
\end{aligned}
$$

Therefore, $\lambda_k \stackrel{\text{def}}{=} \tilde{\lambda}_k - ||r||^2$ is an eigenvalue of Hill's operator L_q and item (a) of Lemma 1 is proved.

Item (b) follows from the previous arguments together with the invertibility of the map $\Phi_\rho : H^1(\mathbb{T}^1_2) \to H^1(\mathbb{T}^1_2)$ defined as multiplication by $1/\rho$.

Item (c) follows from item (a) and the fact that $\tilde{\lambda}_0(r) = 0$ is the first eigenvalue of the impedance operator T_r (see Appendix A, §5.1).

To prove item (d), note that $\tilde{\lambda}_0(r) = 0$ is a simple eigenvalue of T_r and $\tilde{f}_0 \equiv 1/||\rho||$ a corresponding eigenfunction – see Appendix A, §5.1, for a proof. By item (b), $\lambda_0(q)$ is simple as well. The corresponding eigenspace is spanned by the function $f_0 \stackrel{\text{def}}{=} \Phi_\rho^{-1}(1/||\rho||) = \rho/||\rho||$ which is in $H^1(\mathbb{T}^1)$ and does not vanish on $\mathbb{T}^1$. In particular, $\frac{f_0'}{f_0} = \frac{\rho'}{\rho}$ and as ρ satisfies $r = \rho'/\rho$ it follows that $r = f_0'/f_0$, which finishes the proof of (d). $\square$

Lemma 1 (d) implies the following result:

Corollary 1. *The Riccati map $R : L^2_0(\mathbb{T}^1) \to H^{-1}_0(\mathbb{T}^1)$ is injective. Moreover, if $q \in \texttt{range}(R)$ then $R^{-1}(q) = f_0'(\cdot, q)/f_0(\cdot, q)$.*

Remark 1. *Note that the quotient f_0'/f_0 is independent of the normalization of f_0.*

By the Sobolev embedding $H^1(\mathbb{T}^1_2) \hookrightarrow C^0(\mathbb{T}^1_2)$ one concludes that for any $u \in H^0(\mathbb{T}^1_2)$ and $v \in H^1(\mathbb{T}^1_2)$, $uv \in H^0(\mathbb{T}^1_2)$. The products $u'v$ and uv' are elements in $H^{-1}(\mathbb{T}^1_2)$ defined by the dual pairing $\langle \cdot, \cdot \rangle$, i.e. for any $\phi \in H^1(\mathbb{T}^1_2)$, $\langle u'v, \phi \rangle \stackrel{\text{def}}{=} \langle u', v\phi \rangle$ and $\langle uv', \phi \rangle \stackrel{\text{def}}{=} (u, v'\phi)$.

In the proof of Lemma 1 we have used the following lemma which can be proved in a straightforward way.

Lemma 2. *Let $u \in H^0(\mathbb{T}^1_2)$ and $v \in H^1(\mathbb{T}^1_2)$. Then the following statements hold:*

(a) $uv \in H^0(\mathbb{T}^1_2)$ *and* $(uv)' = uv' + u'v$*;*

(b) if $v(x) \neq 0$ *for every* $x \in \mathbb{T}^1_2$ *then* $\frac{1}{v} \in H^1(\mathbb{T}^1_2)$ *and* $\left(\frac{1}{v}\right)' = -\frac{v'}{v^2}$.

$H^1(\mathbb{T}^1_2) \hookrightarrow C^0(\mathbb{T}^1_2)$ $C^\infty(\mathbb{T}^1_2)$ with $L^2(\mathbb{T}^1_2)$,

3 Riccati's map

By definition, Riccati's map $R : L^2_0(\mathbb{T}^1) \to H^{-1}_0(\mathbb{T}^1)$ is given by formula (3). For any $\alpha \geq 0$ denote by R_α the restriction of R to $H^\alpha_0(\mathbb{T}^1) \subset L^2_0(\mathbb{T}^1)$, $R_\alpha \stackrel{\text{def}}{=} R|_{H^\alpha_0(\mathbb{T}^1)}$.

Proposition 1. *The Riccati map $R_\alpha : H^\alpha_0(\mathbb{T}^1) \to H^{\alpha-1}_0(\mathbb{T}^1)$ is a diffeomorphism fron $H^\alpha_0(\mathbb{T}^1)$ to $H^{\alpha-1}_0(\mathbb{T}^1)$.*

Proof. Let us consider first the case $\alpha = 0$. By definition, $R_0 = R$. First note that R is continuous. The claimed statement then follows from the following three assertions:

(i) R has dense image;

(ii) R is surjective (and hence bijective by Corollary 1);

(iii) R and R^{-1} are differentiable.

Let $q \in C^\infty_0(\mathbb{T}^1) \stackrel{\text{def}}{=} \{f \in C^\infty(\mathbb{T}^1) | \int_0^1 f(x)dx = 0\}$ and $\lambda_0(q)$ be the first eigenvalue of L_q. By the classical theory of Hill's equation, the corresponding eigenfunction f_0 doesn't have zeroes (see [11]). Hence, $L_q f_0 = \lambda_0(q) f_0$ can be rewritten as $q - \lambda_0(q) = \frac{f_0''}{f_0} = r' + r^2$ where $r \stackrel{\text{def}}{=} \frac{f_0'}{f_0}$. Integrating the last equality one gets $\lambda_0(q) = -||r||^2$, and therefore, $q = r' + r^2 - ||r||^2 = R(r)$. As $C^\infty_0(\mathbb{T}^1)$-functions are dense in $H^{-1}_0(\mathbb{T}^1)$, the image of the Riccati map is dense in $H^{-1}_0(\mathbb{T}^1)$ proving item (i).

To prove item (ii) take an arbitrary $q \in H^{-1}_0(\mathbb{T}^1)$. It follows from item (i) that there exists a sequence $\{q_k\}_{k=1}^\infty \subset \texttt{range}(R)$ such that $q_k \to q$ $(k \to \infty)$ in $H^{-1}_0(\mathbb{T}^1)$. Consider the sequence $\{r_k\}_{k=1}^\infty \subset L^2_0(\mathbb{T}^1)$ such that $q_k = R(r_k)$. Item (c) of Lemma 1 shows that $||r_k||^2 = |\lambda_0(q_k)|$ where $\lambda_0(q_k)$ is the first eigenvalue of the Hill operator $L_{q_k} = -\frac{d^2}{dx^2} + q_k$. As a function of the potential $q \in H^{-1}_0(\mathbb{T}^1)$ the first eigenvalue $\lambda_0(q)$ is continuous on $H^{-1}_0(\mathbb{T}^1)$ – see Appendix B, Lemma 8. Therefore, $||r_k|| \to \sqrt{|\lambda_0(q)|}$ as $k \to \infty$. Hence, the sequence $\{r_k\}_{k=1}^\infty$ is bounded in $H^0(\mathbb{T}^1)$.

Consider the map $S_0 : H^0(\mathbb{T}^1) \to H^{-1}(\mathbb{T}^1)$, $r \mapsto r^2$. This map can be viewed as a composition of two maps $S_0 = \imath \circ S_1$, where $S_1 : H^0(\mathbb{T}^1) \to H^{-3/4}(\mathbb{T}^1)$ is given by the formula $r \mapsto r^2$, and $\imath : H^{-3/4}(\mathbb{T}^1) \hookrightarrow H^{-1}(\mathbb{T}^1)$ is the standard inclusion of Sobolev spaces. By Rellich's theorem $\imath$ is compact. As $|\widehat{(r^2)}_k| \leq ||r||^2$ for any $k \in \mathbb{Z}$ it follows that there exists $C > 0$ so that $||r^2||_{-3/4} \leq C||r||^2$, hence the map S_1 is bounded. Therefore, there exists a subsequence $\{r_{k_j}\}_{j=1}^\infty$ of $\{r_k\}$ and an element $g \in H^{-1}(\mathbb{T}^1)$ such that $r_{k_j}^2 \to g$ $(j \to \infty)$ in $H^{-1}(\mathbb{T}^1)$. By the definition of Riccati's map, $r'_{k_j} = q_{k_j} - r_{k_j}^2 + ||r_{k_j}||^2 \in H_0^{-1}(\mathbb{T}^1)$. Each of the terms on the right hand side of the latter equation converges in $H^{-1}(\mathbb{T}^1)$. Hence r'_{k_j} converges to some element $s \in H_0^{-1}(\mathbb{T}^1)$. Denote by r the unique element in $L_0^2(\mathbb{T}^1)$ such that $s = r'$. As $||r - r_{k_j}|| \leq const\, ||r' - r'_{k_j}||_{-1} \to 0$ for $j \to \infty$ and R is continuous it then follows that $q_{k_j} = R(r_{k_j}) \to R(r)$ $(j \to \infty)$ in $H^{-1}(\mathbb{T}^1)$. Therefore, $q = R(r)$ and claim (ii) is proved.

Towards claim (iii) note that R is continuously differentiable. The corresponding property for R^{-1} follows from the identity $R^{-1}(q) = \frac{f_0'(\cdot,q)}{f_0(\cdot,q)}$ (see Lemma 1 (d)) as follows: the first eigenfunction $f_0(\cdot, q)$ considered as a map $H^{-1}(\mathbb{T}^1) \to H^1(\mathbb{T}^1_2)$ is continuously differentiable (in fact real analytic) in the variable q when normalized so that $\int_0^2 f_0^2(x,q)dx = 2$ and $f_0(0,q) > 0$ (see Appendix B, Lemma 8). Note that any one-periodic function $f \in H^1(\mathbb{T}^1_2)$ can be isometrically identified with a function in $H^1(\mathbb{T}^1)$. Indeed, it follows from $f(x+1) = f(x)$ that the coefficients $\hat{f}_{2k+1}$ $(k \in \mathbb{Z})$ in the Fourier expansion $f = \sum_{k\in\mathbb{Z}} \hat{f}_k e^{i\pi kx}$ of the function f on the interval $[0,2]$ all vanish and the condition $f \in H^1(\mathbb{T}^1_2)$ implies that $f \in H^1(\mathbb{T}^1)$. In particular, we see that the normalized eigenfunction $f_0(\cdot, q)$ considered as a map $H^{-1}(\mathbb{T}^1) \to H^1(\mathbb{T}^1)$ is continuously differentiable (in fact real analytic). This shows that R^{-1} is continuously differentiable.

In order to prove the theorem for arbitrary $\alpha \geq 0$, first note that by Lemma 10 (ii) (Appendix C), $R_\alpha(H^\alpha(\mathbb{T}^1)) \subset H^{\alpha-1}(\mathbb{T}^1)$. To prove that R_α is onto take an arbitrary $q \in H_0^{\alpha-1}(\mathbb{T}^1)$. By the case $\alpha = 0$, $q = R(r)$ for some $r \in L_0^2(\mathbb{T}^1)$. Denote by f_0 the eigenfunction corresponding to the first eigenvalue of the Hill operator L_q on $H^1(\mathbb{T}^1_2)$ normalized so that $\int_0^1 f_0(x,q)^2 dx = 1$ and $f_0(0,q) > 0$. By item (d) of Lemma 1, f_0 is an element of $H^1(\mathbb{T}^1)$, doesn't vanish on $\mathbb{T}^1$, and $r = \frac{f_0'}{f_0}$. Moreover, one concludes from $L_q f_0 = \lambda_0 f_0$ and the fact that the product qf_0 belongs to $H^{\alpha-1}(T^1_2)$ (see Lemma 10, Appendix C) that $f_0 \in H^{\alpha+1}(\mathbb{T}^1_2)$. Using that $f_0(x+1) = f_0(x)$ and arguing as in the case $\alpha = 0$, one concludes that $f_0 \in H^{\alpha+1}(\mathbb{T}^1)$. As $H^{\alpha+1}(\mathbb{T}^1)$ is an algebra it then follows that $1/f_0 \in H^{\alpha+1}(\mathbb{T}^1)$ and $f_0' \in H^\alpha(\mathbb{T}^1)$. Therefore, $r = \frac{f_0'}{f_0} \in H_0^\alpha(\mathbb{T}^1)$. The claim that $R_\alpha^{-1}(q) = \frac{f_0'(q)}{f_0(q)}$ is continuously differentiable follows from the same arguments as in the proof of (iii) for the case $\alpha = 0$ – see Appendix B, Remark 4. $\square$

The following statement is a generalization of the corresponding classical result. Attempts to prove it using the classical approach (see [11]) fail at several stages.

Corollary 2. *For any $q \in H_0^{-1}(\mathbb{T}^1)$, the first eigenvalue $\lambda_0(q)$ of L_q is simple. Any eigenfunction corresponding to $\lambda_0(q)$ is an element in $H^1(\mathbb{T}^1)$ and doesn't vanish on $\mathbb{T}^1$. Moreover $||R^{-1}(q)||$ is a spectral invariant of L_q.*

Proof. Take $r = R^{-1}(q)$. By Lemma 1 (d) the first eigenvalue $\lambda_0(q)$ of L_q is simple and $f_0(x) = \rho(x)/||\rho||$ is an eigenfunction corresponding to $\lambda_0(q)$ where $\rho(x) = \exp(\int_0^x r(v)dv)$. Hence f_0 is an element in $H^1(\mathbb{T}^1)$ and doesn't vanish on $\mathbb{T}^1$. It follows from Lemma 1 (c) that $||R^{-1}(q)||^2 = -\lambda(q)$ which is obviously a spectral invariant. $\square$

Denote by $H_0^\alpha(\mathbb{T}^1, \mathbb{C})$ the complexification of the (real) Sobolev space $H_0^\alpha(\mathbb{T}^1)$. For complex-valued functions $r \in H_0^\alpha(\mathbb{T}^1, \mathbb{C})$, $\alpha \geq 0$, the (complex) Riccati map is defined by the formula $R_\alpha(r) \stackrel{\text{def}}{=} r' + r^2 - \int_0^1 r^2(x)dx$. Using the same arguments as in the real case one concludes that R_α maps $H_0^\alpha(\mathbb{T}^1, \mathbb{C})$ into $H_0^{\alpha-1}(\mathbb{T}^1, \mathbb{C})$ and is an an analytic map. As a consequence of Proposition 1 one obtains

Theorem 2. *For any $\alpha \geq 0$ there exist open neighborhoods $U \subset H_0^\alpha(\mathbb{T}^1, \mathbb{C})$ and $W \subset H_0^{\alpha-1}(\mathbb{T}^1, \mathbb{C})$ of $H_0^\alpha(\mathbb{T}^1)$ and $H_0^{\alpha-1}(\mathbb{T}^1)$ respectively such that the Riccati map $R_\alpha : U \to W$ is an analytic isomorphism.*

Proof. we see that By Proposition 1 it follows that for any $r \in H_0^\alpha(\mathbb{T}^1)$ the complex differential $d_r R_\alpha : H_0^\alpha(\mathbb{T}^1, \mathbb{C}) \to H_0^{\alpha-1}(\mathbb{T}^1, \mathbb{C})$ is a linear isomorphism. Hence for any $r \in H_0^\alpha(\mathbb{T}^1)$ there exist open neighborhoods $U(r)$ of r in $H_0^\alpha(\mathbb{T}^1)$ and $W(q)$ of $q = R_\alpha(r)$ in $H_0^{\alpha-1}(\mathbb{T}^1)$ so that $R_\alpha : U(r) \to W(q)$ is a diffeomorphism.

For q in $H_0^{\alpha-1}(\mathbb{T}^1, \mathbb{C})$, $\alpha \geq 0$, the spectrum of Hill's operator L_q has similar features as in the case where q is real-valued – see [3]. In particular, the spectrum is discrete and of the form $\{\lambda_k(q)\}_{k\geq 0}$ with $\mathtt{Re}(\lambda_k(q)) \leq \mathtt{Re}(\lambda_{k+1}(q))$ and $\mathtt{Re}(\lambda_k(q)) \to \infty$ as $k \to \infty$. Moreover, the simple eigenvalues and corresponding suitably normalized eigenfunctions are locally analytic in q.

Hence, by choosing the neighborhood $U(r)$ introduced above smaller if necessary we may assume in view of Corollary 2 that for any $r \in H_0^\alpha(\mathbb{T}^1)$, the neighborhood $R(U(r))$ of $q = R_\alpha(r)$ has the property that for any $p \in R(U(r))$, the eigenvalue $\lambda_0(p)$ is simple and the corresponding eigenfunction $f_0(\cdot, p)$, normalized by $\int_0^1 f_0(x,p)^2 dx = 1$ and $\mathtt{Re}(f_0(0,p)) > 0$, has the property $\mathtt{Re}(f_0(x,p)) > 0$ for any $x \in \mathbb{T}^1$ and $f_0(\cdot,p) \in H^{\alpha+1}(\mathbb{T}^1)$. Arguing as in the proof of Lemma 1 (d) one then obtains $R^{-1}(p) = f_0'(\cdot,p)/f_0(\cdot,p)$ for any $p \in W(q)$. Hence, $R_\alpha : U \to W$ is a diffeomorphism where $U \stackrel{\text{def}}{=} \bigcup_{r\in H_0^\alpha(\mathbb{T}^1)} U(r) \subset H_0^\alpha(\mathbb{T}^1,\mathbb{C})$ and $W \stackrel{\text{def}}{=} \bigcup_{r\in H_0^\alpha(\mathbb{T}^1)} R_\alpha(U(r)) \subset H_0^{\alpha-1}(\mathbb{T}^1,\mathbb{C})$ are open neighborhoods of $H_0^\alpha(\mathbb{T}^1)$ and $H_0^{\alpha-1}(\mathbb{T}^1)$ respectively. $\square$

Proof of Theorem 1. The claimed results follow from Theorem 2 and Lemma 1. $\square$

4 Applications

This section contains several applications of the Riccati representation of the elements of the Sobolev space $H^{-1}(\mathbb{T}^1)$: we give asymptotic formulas for the spectrum of the Hill operator with a singular potential and an a priori estimate of the potential q in terms of the gap lengths. Analogous asymptotic formulas are proved for the Dirichlet spectrum. Moreover we show that some features of the classical Floquet theory of Hill's operator available till now only in the case of potentials $q \in L^2(\mathbb{T}^1)$ can be extended without essential changes to the case of singular potentials from the Sobolev spaces $H^{-\alpha}(\mathbb{T}^1)$, $0 < \alpha \le 1$.

4.1 Periodic spectrum

In this paragraph we prove four results on the spectrum of Hill's operator.

Theorem 3. *The spectrum of Hill's operator $L_q = -\frac{d^2}{dx^2} + q$ on $H^{-1}(\mathbb{T}^1_2)$ with singular potential $q \in H^{-1}(\mathbb{T}^1)$ is discrete, $\mathtt{spec}(L_q) = \{\lambda_0(q) < \lambda_1(q) \le \lambda_2(q) < ...\}$, $\lambda_k(q) \to \infty$ as $k \to \infty$. The eigenvalues are totally ordered, $\lambda_{2k-1}(q) \le \lambda_{2k}(q)$ and $\lambda_{2k}(q) < \lambda_{2k+1}(q)$, where the equality $\lambda_{2k-1}(q) = \lambda_{2k}(q)$ means that the corresponding eigenspace has two dimensions. Otherwise, the corresponding eigenspaces are one-dimensional. The eigenvalues $\lambda_{2k-1}(q) \le \lambda_{2k}(q)$ with k odd correspond to anti-periodic eigenfunctions, i.e. $f(x+1) = -f(x)$ and those with k even to periodic ones, i.e. $f(x+1) = f(x)$.*

Proof. By Theorem 1 there exists $r \in L_0^2(\mathbb{T}^1)$ such that $q = R(r)$. Theorem 3 then follows from Lemma 1 and the spectral properties of the impedance operator T_r (see Appendix A, §5.1). $\square$

For any $k \geq 0$, denote by $\gamma_k(q) \stackrel{\text{def}}{=} \lambda_{2k}(q) - \lambda_{2k-1}(q)$ the *k'th gap-length* and by $\gamma(q)$ the sequence $\{\gamma_k(q)\}_{k\geq 1}$. The following two theorems are applications of results in [6] concerning the spectrum of the impedance operator T_r for $r \in H_0^0(\mathbb{T}^1)$.

Theorem 4. *For any $q \in H_0^{-1}(\mathbb{T}^1)$, $\{\gamma_k(q)\}_{k\geq 1}$ belongs to the sequence space h^{-1}.*

Theorem 5. *There exists a constant $c > 0$ such that for every potential $q \in H_0^{-1}(\mathbb{T}^1)$*

$$||q||_{-1} \leq c||\gamma(q)||_{-1}(1 + c||\gamma(q)||_{-1})^3 \tag{5}$$

Proof of Theorems 4 and 5. Take $r \stackrel{\text{def}}{=} R^{-1}(q)$. By Lemma 1, the operators L_q and T_r have, up to a translation, the same spectrum. Hence these operators have the same gap-lengths. Theorem 4 thus follows from Theorem 1.1 in [6].

Using $q = R(r) \stackrel{\text{def}}{=} r' + r^2 - ||r||^2$, the Cauchy-Schwartz inequality, and the easily verified inequalities $||r'||_{-1} \leq ||r||$ and $||r^2||_{-1} \leq c_1||r||^2$ for some constant $c_1 > 0$, one concludes that there is a constant $c_2 > 0$ so that for any $r \in L_0^2(\mathbb{T}^1)$, and $q \in H_0^{-1}(\mathbb{T}^1)$ with $q = R(r)$,

$$||q||_{-1} = ||R(r)||_{-1} \leq ||r'||_{-1} + ||r^2||_{-1} + ||r||^2 \leq c_2||r||(1 + c_2||r||). \tag{6}$$

By Theorem 1.2 in [6] for the impedance operator T_r, there exists $c_3 > 0$ so that for any $r \in L_0^2(\mathbb{T}^1)$ and $q \in H_0^{-1}(\mathbb{T}^1)$ with $q = R(r)$

$$||r|| \leq c_3||\gamma(q)||_{-1}(1 + c_3||\gamma(q)||_{-1}). \tag{7}$$

Combining these last two estimates, Theorem 5 follows. $\square$

Remark 2. *By Lemma 1 (d) and (7) we obtain the following estimate of the first eigenvalue $\lambda_0(q)$ in the terms of the sequence of gap lengths*

$$\sqrt{|\lambda_0(q)|} \leq c_3||\gamma(q)||_{-1}(1 + c_3||\gamma(q)||_{-1}).$$

4.2 Dirichlet spectrum

Consider the operator $L_q^{Dir} = -\frac{d^2}{dx^2} + q$ on $H^{-1}[0,1] = (H_c^1[0,1])'$ with $q \in H^{-1}(\mathbb{T}^1)$ and domain $\texttt{Dom}(L_q^{Dir}) = H_{Dir}^1[0,1]$ (see Appendix B, §6.2). First we need some auxiliary results.

Lemma 3. *For given $q \in H_0^{-1}(\mathbb{T}^1)$, let $r \stackrel{\text{def}}{=} R^{-1}(q) \in L_0^2(\mathbb{T}^1)$ where R^{-1} is the inverse of Riccati's map. Then*

(a) $\mathtt{spec}(L_q^{Dir}) = \mathtt{spec}(T_r^{Dir}) - ||r||^2$;

(b) for any $k \geq 1$, the eigenspaces $V_{\mu_k}(L_q^{Dir})$ and $V_{\mu_k + ||r||^2}(T_r^{Dir})$ have the same dimension.

Proof. The proof is the same as the one of Lemma 1. The only difference is that all calculations must be performed in the Sobolev space $H^{-1}[0,1]$ instead of $H^{-1}(\mathbb{T}^1)$. This is possible using Lemma 4 below. $\square$

The following result is a version of Lemma 2 and can be proved in the same way.

Lemma 4. *For any $u \in L^2[0,1]$ and $v \in H^1[0,1]$ the following statements hold:*

(a) $uv \in L^2[0,1]$ and $(uv)' = uv' + u'v$;

(b) if $v(x) \neq 0$ for every $x \in [0,1]$ then $\frac{1}{v} \in H^1[0,1]$ and $\left(\frac{1}{v}\right)' = -\frac{v'}{v^2}$.

The next two theorems generalize results of [13].

Theorem 6. *The spectrum of L_q^{Dir} is discrete $\mathtt{spec}(L_q^{Dir}) = \{\infty < \mu_1(q) < \mu_2(q) < ...\}$, the corresponding eigenspaces are one-dimensional, and $\mu_k \to \infty$ as $k \to \infty$.*

Proof. The claimed results follow from the spectral properties of the operator T_r^{Dir} (see Appendix A, §5.2) and Lemma 3. $\square$

Theorem 7. *A sequence $\{\infty < \sigma_1 < \sigma_2 < ...\}$, $\sigma_k \to \infty$ as $k \to \infty$, is the Dirichlet spectrum of L_q^{Dir} for some $q \in H^{-1}(\mathbb{T}^1)$, if and only if $\sigma_k = const + (k\pi + s_k)^2$ where $\{s_k\}_{k\geq 1} \in h^0$.*

Before proving Theorem 7 let us state Theorem 8. Given $q \in H_0^{-1}(\mathbb{T}^1)$ let $r \stackrel{\text{def}}{=} R^{-1}(q)$. The eigenvalues $\tilde{\mu}_k$ of T_r^{Dir} satisfy $0 < \tilde{\mu}_1 < \tilde{\mu}_2 < ...$ and, according to Lemma 3, $\tilde{\mu}_k = \mu_k + ||r||^2$ where $\mu_{k\geq 1}$ is the spectrum of L_q^{Dir}. Introduce the sequence $\omega(q) \stackrel{\text{def}}{=} \{(\ln(\tilde{\mu}_k^{1/2})\}_{k\geq 1}$. Unlike the eigenfunctions of

L_q^{Dir}, an eigenfunction $\tilde{g}_k$ of the operator T_r^{Dir} corresponding to the eigenvalue $\tilde{\mu}_k = \mu_k + ||r||^2$ is in $H^2[0,1]$. Hence we can define $\kappa(q) \stackrel{\text{def}}{=} \{\kappa_k(q)\}$, $\kappa_k(q) \stackrel{\text{def}}{=} \ln\left|\frac{\tilde{g}_k'(1)}{\tilde{g}_k'(0)}\right|$. Note that $\kappa_k(q)$ is independent of the normalization of $\tilde{g}_k$ and for q in $L_0^2(\mathbb{T}^1)$ coincides with $\ln|g_k'(1)/g_k'(0)|$ where $g_k \in H^2[0,1]$ is an eigenfunction of L_q^{Dir} corresponding to the eigenvalue μ_k. Let us consider the map

$$H_0^{-1}(\mathbb{T}^1) \ni q \mapsto (\omega(q), \kappa(q)). \tag{8}$$

To discuss its properties introduce

$$S \stackrel{\text{def}}{=} \{\{\ln(k\pi + s_k)\}_{k\geq 1} |\ \{s_k\}_{k\geq 1} \in h^0, 0 < k\pi + s_k < (k+1)\pi + s_{k+1}\}.$$

Theorem 8. *The mapping $q \mapsto (\mu(q), \kappa(q))$ is a real analytic isomorphism onto $S \times h^0$.*

Proof of Theorem 7 and 8. The stated results follow directly from Corollary 5.5 and Corollary 5.6 in [2] together with Lemma 3 and Theorem 2. $\square$

4.3 Discriminant of Hill's operator

The potentials $q \in H_0^{-1}(\mathbb{T}^1)$ are too singular for L_q to admit fundamental solutions. Hence the Floquet matrix cannot be defined in this situation. However, it turns out that the trace $\Delta(\lambda, q)$ of the Floquet matrix, often refered to as discriminant, can still be defined as we will explain now. Recall that the discriminant $\tilde{\Delta}$ is defined for $\tilde{\lambda} \in \mathbb{C}$ and $r \in L_0^2(\mathbb{T}^1)$ arbitrary, by $\tilde{\Delta}(\tilde{\lambda}, r) \stackrel{\text{def}}{=} u_1(1, \tilde{\lambda}, r) + u_2'(1, \tilde{\lambda}, r)$ where $u_1(x, \tilde{\lambda}, r)$ and $u_2(x, \tilde{\lambda}, r)$ are the fundamental solutions of the equation

$$-u'' - 2ru' = \tilde{\lambda}u. \tag{9}$$

For $q \in L_0^2(\mathbb{T}^1)$ the discriminant $\Delta(\lambda, q)$ is well defined and related to $\tilde{\Delta}(\tilde{\lambda}, r)$ as follows. Define $r \stackrel{\text{def}}{=} R^{-1}(q)$ and $\rho(x) \stackrel{\text{def}}{=} \exp(\int_0^x r(v)dv)$. As $q \in L^2(\mathbb{T}^1)$, the equation

$$-y'' + qy = \lambda y \tag{10}$$

admits fundamental solutions $y_1(x, \lambda, q)$ and $y_2(x, \lambda, q)$, i.e. solutions satisfying the initial conditions $y_1(0, \lambda, q) = y_2'(1, \lambda, q) = 1$ and $y_1'(0, \lambda, q) =$

$y_2(0,\lambda,q)=0$. By computations as in (4) we see that for $j=1,2$, the functions $\tilde{y}_j(x,\tilde{\lambda},r) \stackrel{\text{def}}{=} y_j(x,\lambda,q)/\rho(x)$ are solutions of (9) with $\tilde{\lambda}=\lambda+||r||^2$ and $||r||^2=\int_0^1 r^2(x)dx$. As $\rho(0)=1$, one obtains $\tilde{y}_1(0,\tilde{\lambda},r)=\tilde{y}_2'(0,\tilde{\lambda},r)=1$, $\tilde{y}_2(0,\tilde{\lambda},r)=0$, and $\tilde{y}_1'(0,\tilde{\lambda},q)=-\rho'(0)$. Hence, the fundamental solutions u_1 and u_2 of equation (9) are related to $\tilde{y}_1$ and $\tilde{y}_2$ by

$$u_1(x,\tilde{\lambda},r)=\tilde{y}_1(x,\tilde{\lambda},r)+\rho'(0)\tilde{y}_2(x,\tilde{\lambda},r)$$

and

$$u_2(x,\tilde{\lambda},r)=\tilde{y}_2(x,\tilde{\lambda},r).$$

Using that $\rho(1)=\rho(0)=1$ and $\rho'(0)=\rho'(1)$ we obtain

$$\begin{aligned}\tilde{\Delta}(\tilde{\lambda},r) &\stackrel{\text{def}}{=} u_1(1,\tilde{\lambda},r)+u_2'(1,\tilde{\lambda},r)\\ &= (y_1/\rho+\rho'(0)y_2/\rho)|_{(x=1,\lambda,q)}+(y_2'/\rho-y_2\rho'/\rho^2)|_{(x=1,\lambda,q)}\\ &= y_1(1,\lambda,q)+y_2'(1,\lambda,q) \stackrel{\text{def}}{=} \Delta(\lambda,q).\end{aligned} \tag{11}$$

Hence, we can *define* $\Delta(\lambda,q)$ for $q\in H_0^{-1}(\mathbb{T}^1)$ by the latter identity.

Definition 1. *For any $q\in H_0^{-1}(\mathbb{T}^1)$ and $\lambda\in\mathbb{C}$,*

$$\Delta(\lambda,q) \stackrel{\text{def}}{=} \tilde{\Delta}(\lambda+||r||^2,r) \tag{12}$$

where $r=R^{-1}(q)$, $||r||^2=\int_0^1 r^2(x)dx$.

It follows directly from Theorem 2 and the properties of $\tilde{\Delta}(\tilde{\lambda},r)$ that $\Delta(\lambda,q)$ is an analytic function on $\mathbb{C}\times W$ where $W\subset H_0^{-1}(\mathbb{T}^1,\mathbb{C})$ is the neighborhood of $H_0^{-1}(\mathbb{T}^1)$ given by Theorem 2, and that the zeroes of $\Delta(\lambda,q)^2-4$ are precisely the eigenvalues of L_q, i.e. for any $\lambda\in\mathbb{C}$,

$$\Delta(\lambda,q)^2-4=0 \textit{ if and only if } \lambda\in\mathtt{spec}(L_q).$$

4.4 Isospectral invariance of Riccati's map

For any $q\in H_0^{-1}(\mathbb{T}^1)$ denote by $\mathtt{Iso}(L_q)$ the set of potentials $p\in H_0^{-1}(\mathbb{T}^1)$ such that $\mathtt{spec}(L_p)=\mathtt{spec}(L_q)$, i.e.

$$\mathtt{Iso}(L_q) \stackrel{\text{def}}{=} \{p\in H_0^{-1}(\mathbb{T}^1)|\,\mathtt{spec}(L_p)=\mathtt{spec}(L_q)\}.$$

Similarly for any $r\in L_0^2(\mathbb{T}^1)$, denote

$$\mathtt{Iso}(T_r) \stackrel{\text{def}}{=} \{u\in L_0^2(\mathbb{T}^1)|\,\mathtt{spec}(T_u)=\mathtt{spec}(T_r)\}.$$

Theorem 9. *For every $r \in L_0^2(\mathbb{T}^1)$*

$$R(\mathtt{Iso}(T_r)) = \mathtt{Iso}(L_{R(r)}).$$

Proof. Let $r \in L_0^2(\mathbb{T}^1)$ and $q \stackrel{\text{def}}{=} R(r) \in H_0^{-1}(\mathbb{T}^1)$. To see that $\mathtt{Iso}(L_{R(r)}) \subset R(\mathtt{Iso}(T_r))$ take any $p \in \mathtt{Iso}(L_q)$ and set $u \stackrel{\text{def}}{=} R^{-1}(p)$. By Lemma 1 (a) and (d),

$$\mathtt{spec}(T_u) = -\lambda_0(p) + \mathtt{spec}(L_p) = -\lambda_0(q) + \mathtt{spec}(L_q) = \mathtt{spec}(T_r).$$

Conversely, take $u \in \mathtt{Iso}(T_r)$ and let $p \stackrel{\text{def}}{=} R(u)$. By the definition of the isospectral set $\mathtt{Iso}(T_r)$ we obtain that $\mathtt{spec}(T_u) = \mathtt{spec}(T_r)$. It follows from Lemma 1 (a) that $\mathtt{spec}(L_p) = \mathtt{spec}(T_u) - ||u||^2$ and $\mathtt{spec}(L_q) = \mathtt{spec}(T_r) - ||r||^2$. By Corollary 4 (Appendix A) the L^2-norm $||r||$ of the potential $r \in L_0^2(\mathbb{T}^1)$ is a spectral invariant of the impedance operator T_r. Hence, it follows from $\mathtt{spec}(T_u) = \mathtt{spec}(T_r)$, that $||u||^2 = ||r||^2$ and therefore $\mathtt{spec}(L_p) = \mathtt{spec}(L_q)$. This completes the proof of Theorem 9. ☐

Corollary 3. *For every potential $q \in H_0^{-1}(\mathbb{T}^1)$, the isospectral set $\mathtt{Iso}(L_q)$ is compact in $H_0^{-1}(\mathbb{T}^1)$.*

Proof. First we prove that for any $r \in L_0^2(\mathbb{T}^1)$ the isospectral set $\mathtt{Iso}(T_r)$ is compact. Let $\{r_k\}_{k\geq 0}$ be a sequence in $\mathtt{Iso}(T_r)$. It follows from the spectral invariance of the L^2-norm of r that $||r_k|| = ||r||$ for any $k \geq 1$ (Corollary 4). Hence, there exists a subsequence $\{r_{k_j}\}_{j\geq 1}$ that converges weakly to some element $u \in L_0^2(\mathbb{T}^1)$. By Lemma 2.4 in [6], the sequence of discriminants $\tilde{\Delta}(\tilde{\lambda}, r_k)$ of T_{r_k} converges to $\tilde{\Delta}(\tilde{\lambda}, u)$ as $k \to \infty$ uniformly on bounded subsets of $\mathbb{C}$. On the other side, the spectral invariance of r_k shows that $\tilde{\Delta}(\tilde{\lambda}, r_k) = \tilde{\Delta}(\tilde{\lambda}, r)$ and therefore $\tilde{\Delta}(\tilde{\lambda}, r) = \tilde{\Delta}(\tilde{\lambda}, u)$. In particular, $\mathtt{spec}(T_r) = \mathtt{spec}(T_u)$ and $||u|| = ||r||$. As r_{k_j} converges weakly to u, $||u - r_{k_j}||^2 = 2||u||^2 - 2(u, r_{k_j}) \to 0$ as $j \to \infty$. Therefore, the isospectral set $\mathtt{Iso}(T_r)$ is compact.

Theorem 9 and the continuity of the Riccati map $R : L_0^2(\mathbb{T}^1) \to H^{-1}(\mathbb{T}^1)$ then imply the compactness of the isospectral sets $\mathtt{Iso}(L_q)$, $q \in H_0^{-1}(\mathbb{T}^1)$. ☐

4.5 Complex potentials

In a straightforward way many of the previous results can be extended for complex potentials in some open neighborhood W of $H_0^{-1}(\mathbb{T}^1)$ in $H_0^{-1}(\mathbb{T}^1, \mathbb{C})$. As an example we mention the following theorem which can be proved using the same arguments as in the proof of Lemma 1.

Denote by U and W the neighborhoods given by Theorem 2.

Theorem 10. *For given $q \in W \subset H_0^{-1}(\mathbb{T}^1, \mathbb{C})$, let $r \stackrel{\text{def}}{=} R^{-1}(q) \in U \subset L_0^2(\mathbb{T}^1, \mathbb{C})$ where R^{-1} is the inverse of Riccati's map $R : U \to W$. Then*

(a) $\texttt{spec}(L_q) = \texttt{spec}(T_r) - \int_0^1 r(x)^2 dx$;

(b) for any $k \geq 1$, the eigenspaces $V_{\lambda_k}(L_q)$ and $V_{\tilde{\lambda}_k}(T_r)$ have the same dimension where $\tilde{\lambda}_k \stackrel{\text{def}}{=} \lambda_k + \int_0^1 r(x)^2 dx$.

In view of Theorem 10 one can reformulate results on the spectrum of the impedance operator T_r with $r \in U$ in terms of the corresponding result for the operator L_q with $q = R(r) \in W$, and vice versa.

5 Appendix A: Impedance operator

An impedance operator is a Sturm-Liouville operator of a special type and is treated in numerous articles and books – see [9, 1, 2, 6]. For the convenience of the reader we recall its properties needed in the main part of this paper.

5.1 Periodic problem

For any $r \in L_0^2(\mathbb{T}^1)$ denote by ρ the element in $H^1(\mathbb{T}^1)$ satisfying $\rho' = r\rho$ and $\rho(0) = 1$. Then ρ is a one-periodic, absolutely continuous, positive function given by $\rho(x) = \exp(\int_0^x r(v)dv)$. The periodic impedance operator T_r is defined on the Hilbert space $L^2(\mathbb{T}_2^1)$ with domain $\texttt{Dom}(T_r) = H^2(\mathbb{T}_2^1)$ by the formula

$$T_r(u) \stackrel{\text{def}}{=} -(\rho^2 u')'/\rho^2 = -u'' - 2ru'. \tag{13}$$

Note that the operator T_r is positive and symmetric with respect to the $L^2(\mathbb{T}_2^1)$-inner product $(f, g)_\rho \stackrel{\text{def}}{=} \int_0^2 fg\rho^2 dx$.

As T_r has compact resolvent the spectrum of T_r is discrete. It turns out (see [9]) that $\texttt{spec}(T_r)$ is of the form $\texttt{spec}(T_r) = \{0 = \tilde{\lambda}_0(r) < \tilde{\lambda}_1(r) \leq$

$\tilde\lambda_2(r) \le ...\}$, the corresponding eigenspaces are of dimension 1 or 2, and $\tilde\lambda_k(r) \to \infty$ as $k \to \infty$. For $k \ge 0$ even, the eigenfunctions $\tilde f_{2k-1}$ and $\tilde f_{2k}$ are periodic while for k odd, $\tilde f_{2k-1}$ and $\tilde f_{2k}$ are anti-periodic.

Lemma 5. *The first eigenvalue $\tilde\lambda_0(r) = 0$ of T_r is simple and the corresponding eigenspace is spanned by the constant function $\tilde f_0 = 1/||\rho||$.*

Proof. Assume that $u \in \ker(T_r)$. Integrating by parts we obtain

$$0 = (T_r(u), u)_\rho \stackrel{\text{def}}{=} -\int_0^2 (\rho^2 u')' u dx = \int_0^2 \rho^2 (u')^2 dx. \tag{14}$$

As ρ is positive, $u' \equiv 0$ and hence u is constant. This shows that the dimension of $\ker(T_r)$ is equal to one and Lemma 5 is proved. $\square$

The discriminant $\tilde\Delta$ of the impedance operator T_r is defined for $\tilde\lambda \in \mathbb{C}$ and $r \in L_0^2(\mathbb{T}^1)$ arbitrary, by $\tilde\Delta(\tilde\lambda, r) \stackrel{\text{def}}{=} u_1(1, \tilde\lambda, r) + u_2'(1, \tilde\lambda, r)$ where $u_1(x, \tilde\lambda, r)$ and $u_2(x, \tilde\lambda, r)$ are the fundamental solutions of the equation $-u'' - 2ru' = \tilde\lambda u$. It follows from the results in [1] that the discriminant $\tilde\Delta(\tilde\lambda, r)$ is an analytic function on $\mathbb{C} \times L_0^2(\mathbb{T}^1)$. The following lemma can be proved in straightforward way using the results in [1].

Lemma 6. *The discriminant $\tilde\Delta(\tilde\lambda, r)$ is a spectral invariant of the impedance operator T_r. The set of zeroes $\tilde\lambda_k$ of the equation $\tilde\Delta(\tilde\lambda, r)^2 = 4$, counted with their multiplicities, coincides with the spectrum of T_r.*

Lemma 6 together with Corollary 1.2 in [7] then lead to the following

Corollary 4. *The L^2-norm $||r||$ of $r \in L_0^2(\mathbb{T}^1)$ is a spectral invariant of the impedance operator T_r.*

5.2 Dirichlet problem

The Dirichlet problem for the impedance operator has been considered by many authors – see e.g. [9, 1, 2]. The operator T_r^{Dir} is defined on $L^2[0, 1]$ with domain $\texttt{Dom}(T_r^{Dir}) = H^2_{Dir}[0, 1] \stackrel{\text{def}}{=} \{f \in H^2[0, 1] \,|f(0) = f(1) = 0\}$. By definition, the operator T_r^{Dir} acts on elements $u \in H^2_{Dir}[0, 1]$ by the formula $T_r^{Dir}(u) \stackrel{\text{def}}{=} -(\rho^2 u')'/\rho^2 = -u'' - 2ru'$ where $\rho(x) \stackrel{\text{def}}{=} \exp(\int_0^x r(v)dv)$. The spectrum $\texttt{spec}(T_r^{Dir})$ is called *Dirichlet spectrum* of T_r. It is known that

$\mathtt{spec}(T_r^{Dir})$ is discrete, all eigenvalues are simple, and $\mathtt{spec}(T_r^{Dir}) = \{0 < \mu_1(r) < \mu_2(r) < ...\}$ where $\mu_k(r) \to \infty$ as $k \to \infty$. Other spectral properties of T_r^{Dir}, including the solution of an inverse problem, were established in [1, 2]. We refer the reader to these papers.

6 Appendix B: Schrödinger operator

6.1 Periodic problem

Take $q \in H^{-1}(\mathbb{T}^1) \stackrel{\text{def}}{=} (H^1(\mathbb{T}^1))'$ and consider Hill's operator

$$L_q = -\frac{d^2}{dx^2} + q \tag{15}$$

on $H^{-1}(\mathbb{T}^1_2)$ with domain $\mathtt{Dom}(L_q) = H^1(\mathbb{T}^1_2)$. The elements of $H^{-1}(\mathbb{T}^1)$ can be considered as elements of $H^{-1}(\mathbb{T}^1_2)$ as follows: to any element $u \in H^{-1}(\mathbb{T}^1)$ with Fourier expansion $u = \sum_{k\in\mathbb{Z}} \hat{u}_k e^{i2k\pi x}$, we assign the unique element in $H^{-1}(\mathbb{T}^1_2)$ given by the Fourier series $\sum_{k\in\mathbb{Z}} \hat{s}_k e^{ik\pi x}$ with $\hat{s}_{2k} \stackrel{\text{def}}{=} \hat{u}_k$ and $\hat{s}_{2k+1} \stackrel{\text{def}}{=} 0$. The operator L_q acts on elements $u \in H^1(\mathbb{T}^1_2)$ by $L_q u \stackrel{\text{def}}{=} -u'' + qu$ where the multiplication qu is viewed as an element of $H^{-1}(\mathbb{T}^1_2)$ according to the formula $\langle qu, v\rangle \stackrel{\text{def}}{=} \langle q, uv\rangle$. The brackets $\langle\cdot,\cdot\rangle$ denote the dual pairing between the elements of $H^{-1}(\mathbb{T}^1_2) \stackrel{\text{def}}{=} (H^1(\mathbb{T}^1_2))'$ and $H^1(\mathbb{T}^1_2)$. As the multiplication map $H^1(\mathbb{T}^1_2) \times H^1(\mathbb{T}^1_2) \to H^1(\mathbb{T}^1_2)$ given by $u \cdot v \stackrel{\text{def}}{=} uv$ is continuous the linear functional qu is continuous as well. It can be easily seen that L_q induces a bounded operator $L_q : H^1(\mathbb{T}^1_2) \to H^{-1}(\mathbb{T}^1_2)$. Considered as an operator on $H^{-1}(\mathbb{T}^1_2)$, with domain $\mathtt{Dom}(L_q) = H^1(\mathbb{T}^1_2)$, L_q is an unbounded operator.

The operators L_q with singular potentials $q \in H^{-\alpha}(\mathbb{T}^1_2)$, $0 < \alpha \le 1$, have been considered in [12, 3]. In order to make this paper self-contained we review the case $\alpha = 1$ treated in [12] and prove the auxiliary facts used in the main part of the paper. In our presentation, we mainly follow [12], §1.5.1.

For $M > 0$, $r > 0$, and $n \in \mathbb{N}$ introduce the sets

$$Ext_M \stackrel{\text{def}}{=} \{\lambda \in \mathbb{C} \mid \mathtt{Re}(\lambda) \le |\mathtt{Im}(\lambda)| - M\} \tag{16}$$

$$Vert_n(r) \stackrel{\text{def}}{=} \{\lambda = n^2\pi^2 + z \in \mathbb{C} \mid |\mathtt{Re}(z)| \le n\pi^2, |z| \ge r\}. \tag{17}$$

Via the Fourier transform, we identify L_q with the operator $\hat{L}_v$ on h^{-1} with domain $\mathtt{Dom}(\hat{L}_v) = h^1$. The operator $\hat{L}_v$ acts on the sequences $x = \{x_k\}_{k\in\mathbb{Z}} \in$

h^1 by $D+V$ where D and V are the infinite matrices $D \stackrel{\text{def}}{=} (k^2\pi^2\delta_{kl})_{k,l\in\mathbb{Z}}$ and $V \stackrel{\text{def}}{=} (v(k-l))_{k,l\in\mathbb{Z}}$, δ_{kl} is the Kronecker delta and $v(k)$ are the Fourier coefficients of the potential $q \in H^{-1}(\mathbb{T}^1)$ viewed as an element in $H^{-1}(\mathbb{T}^1_2)$. The proof of the following result can be found in [12].

Lemma 7. *For every $v \in h^{-1}$ there exist a neighborhood $U(v) \subset h^{-1}$ of v and constants $M > 0$ and $n_0 \in \mathbb{N}$ such that for every $u \in U(v)$, the sets Ext_M and $Vert_n(n)$ $(n > n_0)$ are contained in the resolvent set $\mathtt{resol}(\hat{L}_u)$ of $\hat{L}_u$. The resolvent $(\lambda\mathbf{1} - \hat{L}_u)^{-1} \in \mathcal{L}(h^{-1}, h^1)$, considered as a function of (λ, u) on $Ext_M \times U(v)$ or $Vert_n(n) \times U(v)$ with $n > n_0$, is continuous in (λ, u) and for every $u \in U(v)$ holomorphic in λ. Moreover, for any smooth contour $\Gamma \subset Ext_M \cup \bigcup_{n>n_0} Vert_n(n)$ and integer $l \geq 0$, the operator $Q^l_\Gamma(q) \stackrel{\text{def}}{=} \frac{1}{2\pi i}\oint_\Gamma \lambda^l(\lambda\mathbf{1} - \hat{L}_u)^{-1} d\lambda \in \mathcal{L}(h^{-1}, h^1)$ is analytic when viewed as a map $U(v) \to \mathcal{L}(h^{-1}, h^1)$.*

Proof. Take an arbitrary $v \in h^{-1}$. Given any $\epsilon > 0$, to be chosen later at our convenience, fix $v_0 \in h^1$ with $||v - v_0||_{-1} < \epsilon$.

For any $u \in U_\epsilon(v_0) \stackrel{\text{def}}{=} \{u \in h^{-1} |\ ||u - v_0||_{-1} < \epsilon\}$, define $v_1 \stackrel{\text{def}}{=} u - v_0$. Then the operator $\hat{L}_u$ can be rewritten as $\hat{L}_u = D + V_0 + V_1$ where the operators V_0 and V_1 are given by the infinite matrices $(v_0(k-l))_{k,l\in\mathbb{Z}}$ and $(v_1(k-l))_{k,l\in\mathbb{Z}}$ respectively. Let $Q_\lambda \stackrel{\text{def}}{=} \lambda\mathbf{1} - D - V_0$.

As a first step we prove that one can choose $M > 0$ and $\epsilon > 0$ so that for any $u \in U_\epsilon(v_0) \subset h^{-1}$ the resolvent $(\lambda\mathbf{1} - \hat{L}_u)^{-1} = (Q_\lambda - V_1)^{-1}$, can be represented in the form

$$\begin{aligned}(\lambda\mathbf{1} - \hat{L}_u)^{-1} &= (Q_\lambda - V_1)^{-1} \\ &= Q_\lambda^{-1}(\mathbf{1} - (V_1 Q_\lambda^{-1}))^{-1} \\ &= Q_\lambda^{-1}\sum_{k\geq 0}(V_1 Q_\lambda^{-1})^k \qquad (18)\end{aligned}$$

where the series in (18) converges uniformly for every $\lambda \in Ext_M$ and $u \in U_\epsilon(v_0)$. Note that the a priori chosen potential $v \in h^{-1}$ belongs to $U_\epsilon(v_0)$.

To prove that $M > 0$ and $\epsilon > 0$ as above exist observe that for any $\lambda \in Ext_M$ with $M > 0$ arbitrary one can write at least formally

$$\begin{aligned}Q_\lambda^{-1} &= ((\lambda\mathbf{1} - D) - V_0)^{-1} = (\lambda\mathbf{1} - D)^{-1}(\mathbf{1} - V_0(\lambda\mathbf{1} - D)^{-1})^{-1} \\ &= (\lambda\mathbf{1} - D)^{-1}\sum_{k\geq 0}(V_0(\lambda\mathbf{1} - D)^{-1})^k \qquad (19)\end{aligned}$$

A direct computation shows that if $M > \pi^2$ one has $||(\lambda\mathbf{1} - D)^{-1}||_{\mathcal{L}(h^{-1},h^1)} < 2$ and $||(\lambda\mathbf{1} - D)^{-1}||_{\mathcal{L}(h^{-1})} \leq \frac{\sqrt{2}}{M}$. By Lemma 10 (Appendix C), there exists a universal constant $c > 0$ so that $||V_0||_{\mathcal{L}(h^{-1})} \leq c||v_0||_1$. Hence,

$$\begin{aligned}||V_0(\lambda\mathbf{1} - D)^{-1}||_{\mathcal{L}(h^{-1})} &\leq ||V_0||_{\mathcal{L}(h^{-1})}||(\lambda\mathbf{1} - D)^{-1}||_{\mathcal{L}(h^{-1})} \\ &\leq c||v_0||_1 \frac{\sqrt{2}}{M}.\end{aligned}$$

Taking $M > 4c||v_0||_1 + \pi^2$, the formal expression (19) converges uniformly in $\lambda \in Ext_M$, and Q_λ is bounded as a map from h^{-1} to h^1 with $||Q_\lambda^{-1}||_{\mathcal{L}(h^{-1},h^1)} \leq 4$ uniformly for $\lambda \in Ext_M$.

Using Lemma 10 (Appendix C),

$$\begin{aligned}||V_1 Q_\lambda^{-1}||_{\mathcal{L}(h^{-1})} &\leq ||V_1||_{\mathcal{L}(h^1,h^{-1})}||Q_\lambda^{-1}||_{\mathcal{L}(h^{-1},h^1)} \\ &\leq 4c_1||v_1||_{-1} \end{aligned} \tag{20}$$

where c_1 is a universal constant, in particular *independent* of $||v_0||_1$. Hence, taking $\epsilon < \frac{1}{8c_1}$ and then choosing $v_0 \in h^1$ with $||v - v_0||_{-1} < \epsilon$, one concludes that for any $M > 4c||v_0||_1 + \pi^2$, $Q_\lambda^{-1} : h^{-1} \to h^1$ is uniformly bounded for $\lambda \in Ext_M$ and the series $\sum_{k\geq 0}(V_1 Q_\lambda^{-1})^k$ converges in $\mathcal{L}(h^{-1})$ uniformly in $(\lambda, u) \in Ext_M \times U_\epsilon(v_0)$. By Lemma 10 (Appendix C), the map $v_1 \mapsto V_1$ is continuous as a map from h^{-1} to $\mathcal{L}(h^1, h^{-1})$. In particular, the expressions $Q_\lambda^{-1}(V_1 Q_\lambda^{-1})^k$ are continuous as functions of the arguments (λ, u), and therefore, the resolvent $(\lambda\mathbf{1} - \hat{L}_u)^{-1} \in \mathcal{L}(h^{-1}, h^1)$ is continuous in $(\lambda, u) \in Ext_M \times U_\epsilon(v_0)$ as well.

The case $(\lambda, u) \in Vert_n(n) \times U_\epsilon(v_0)$ can be treated in the same way. The only difference is that instead of the representation (18), one considers

$$\begin{aligned}(\lambda\mathbf{1} - \hat{L}_u)^{-1} &= \mathbf{1}_{(h^{1,-n},h^1)} Q_\lambda^{-1}\mathbf{1}_{(h^{-1},h^{-1,n})} \\ &+ \mathbf{1}_{(h^{1,-n},h^1)} Q_\lambda^{-1} T_\lambda \mathbf{1}_{(h^{-1,-n},h^{-1,n})}(V_1 Q_\lambda^{-1})\mathbf{1}_{(h^{-1},h^{-1,n})} \\ &+ \mathbf{1}_{(h^{1,-n},h^1)} Q_\lambda^{-1} T_\lambda (V_1 Q_\lambda^{-1})^2 \mathbf{1}_{(h^{-1},h^{-1,n})} \end{aligned} \tag{21}$$

where $\mathbf{1}_{(h^{\pm1,\sigma_1},h^{\pm1,\sigma_2})}$ denotes the "identity operator" considered as a linear map $h^{\pm1,\sigma_1} \to h^{\pm1,\sigma_2}$, $Q_\lambda^{-1} \in \mathcal{L}(h^{-1,l}, h^{1,-l})$ $(l = \pm n)$ is given by

$$Q_\lambda^{-1} \stackrel{\text{def}}{=} (\lambda\mathbf{1} - D)^{-1}\sum_{k\geq 0}(V_0(\lambda\mathbf{1} - D)^{-1})^k \tag{22}$$

with

$$(\lambda\mathbf{1} - D)^{-1} : h^{-1,l} \to h^{1,-l}$$

and

$$V_0 : h^{-1,l} \to h^{-1,l}$$

and $T_\lambda \in \mathcal{L}(h^{-1,n})$ is given by

$$T_\lambda \stackrel{\text{def}}{=} \sum_{k\geq 0} (V_1 Q_\lambda^{-1} V_1 Q_\lambda^{-1})^k, \tag{23}$$

where $V_1 Q_\lambda^{-1} V_1 Q_\lambda^{-1}$ is viewed as a composition $h^{-1,n} \to h^{1,-n} \to h^{-1,-n} \to h^{1,n} \to h^{-1,n}$. By Lemma 10 (Appendix C), the operators V_0 and V_1 are bounded operators in $\mathcal{L}(h^{-1,l})$ and $\mathcal{L}(h^{1,l}, h^{-1,l})$ ($l = \pm n$) respectively.

A direct computation shows that there exist constants m_1 and m_2 such that for any $l \in \mathbb{Z} \setminus \{0\}$, $||(\lambda \mathbf{1} - D)^{-1}||_{\mathcal{L}(h^{-1,l}, h^{1,-l})} < m_1$ and $||(\lambda \mathbf{1} - D)^{-1}||_{\mathcal{L}(h^{-1,l})} < m_2/l$. Taking $n_0 \in \mathbb{N}$ sufficiently big, we see that the series in (22) converges to the operator $Q_\lambda^{-1} \in \mathcal{L}(h^{-1,l}, h^{1,-l})$ ($l = \pm n$) and that Q_λ^{-1} is uniformly bounded on $Vert_n(n)$, $n > n_0$. We thus obtain for appropriate constants $C_1 > 0$, $C_2 > 0$, independent of n,

$$||V_1 Q_\lambda^{-1} V_1 Q_\lambda^{-1}||_{\mathcal{L}(h^{-1,n})} \leq C_1 ||V_1||_{\mathcal{L}(h^{1,n}, h^{-1,n})} ||V_1||_{\mathcal{L}(h^{1,-n}, h^{-1,-n})} \leq C_2 ||v_1||_{-1}^2.$$

Hence for $||v_1||_{-1}$ sufficiently small we see that (23) converges absolutely and uniformly on $Vert_n(n) \times U_\epsilon(v_0)$ for any $n > n_0$ to an operator-valued function, continuous in (λ, u).

Finally, taking $\epsilon < \min\{\frac{1}{8c_1}, \frac{1}{\sqrt{2C_2}}\}$ and then choosing v_0 as above, we see that the representations (18) and (21) of the resolvent $(\lambda \mathbf{1} - \hat{L}_u)^{-1} \in \mathcal{L}(h^{-1}, h^1)$ converge uniformly on $Ext_M \times U_\epsilon(v_0)$ and $Vert_n(n) \times U_\epsilon(v_0)$ with $n > n_0$ respectively.

Any smooth and sufficiently regular contour $\Gamma \subset Ext_M \cup \bigcup_{n>n_0} Vert_n(n)$ is a disjoint union of arcs $\Gamma = \Gamma_1 \sqcup ... \sqcup \Gamma_m$ such that $\Gamma_k \subset Ext_M$ or $\Gamma_k \subset \bigcup_{n>n_0} Vert_n(n)$. Hence, $Q_\Gamma^l(q) \stackrel{\text{def}}{=} \frac{1}{2\pi i} \oint_\Gamma \lambda^l (\lambda \mathbf{1} - \hat{L}_u)^{-1} d\lambda = Q_{\Gamma_1}^l(q) + ... + Q_{\Gamma_m}^l(q)$ where $Q_{\Gamma_k}^l(q) \stackrel{\text{def}}{=} \frac{1}{2\pi i} \int_{\Gamma_k} \lambda^l (\lambda \mathbf{1} - \hat{L}_u)^{-1} d\lambda$. It follows from the uniform convergence of the representations (18) and (21) that any $Q_{\Gamma_k}^l(q) \in \mathcal{L}(h^{-1}, h^1)$ is analytic with respect to $q \in U_\epsilon(v_0)$. This proves Lemma 7. $\square$

Lemma 8.

(i) For any $q \in H^{-1}(\mathbb{T}^1)$, L_q has a compact resolvent.

(ii) *The spectrum of the Hill operator L_q with potential $q \in H^{-1}(\mathbb{T}^1)$ is discrete,* $\texttt{spec}(L_q) = \{\lambda_0(q) \le \lambda_1(q) \le \lambda_2(q) \le ...\}$*, the corresponding eigenspaces are of finite dimension, and $\lambda_k(q) \to \infty$ as $k \to \infty$.*

(iii) *As functions of the potential $q \in H^{-1}(\mathbb{T}^1)$, the k'th eigenvalue $\lambda_k(q) : H^{-1}(\mathbb{T}^1) \to \mathbb{R}$, $q \mapsto \lambda_k(q)$ is continuous.*

(iv) *Suppose that the eigenvalue $\lambda_k(q)$ is simple for some $q \in H^{-1}(\mathbb{T}^1)$. Then there exists a neighborhood $U(q) \subset H^{-1}(\mathbb{T}^1)$ of q such that for any $u \in U(q)$, the k-th eigenvalue $\lambda_k(u)$ is simple and the corresponding eigenfunction $f_k(\cdot, u)$ (normalized so that $\int_0^2 f_k^2(x,u)dx = 2$ and $f(x_0, u) > 0$, where $x_0 \in [0,2]$ is chosen so that $f_k(x_0, q) > 0$ for the given potential q) is analytic as a map $U(q) \to H^1(\mathbb{T}^1_2)$.*

Remark 3. *We improve on Lemma 8 in Section 4 (see Theorem 3).*

Remark 4. *An analogue of Lemma 8 is true for potentials $q \in H^{\alpha-1}(\mathbb{T}^1)$ with arbitrary $\alpha \ge 0$. In particular, the eigenvalues $\lambda_k(q)$ are continuous with respect to the norm in $H^{\alpha-1}(\mathbb{T}^1)$. If $\lambda_k(q)$ is simple, the corresponding normalized eigenfunction $f_k(\cdot, q)$ is analytic as a map from a neighborhood of q in $H^{\alpha-1}(\mathbb{T}^1)$ to $H^{\alpha+1}(\mathbb{T}^1_2)$. The proofs are the same as in the case $\alpha = 0$.*

Proof of Lemma 8. We prove a version of Lemma 8 in sequence spaces. In particular we identify L_q with $\hat{L}_v$ – see explanations above.

Let us first prove item (i) and (ii). By Lemma 7 there exists for every $v \in h^{-1}$ a point $z_0 \in \mathbb{C}$ such that the resolvent $(z_0\mathbf{1} - \hat{L}_v)^{-1} \in \mathcal{L}(h^{-1})$ is bounded as a map from h^{-1} to h^1. In particular, by Rellich's theorem, the resolvent $(z_0\mathbf{1} - \hat{L}_v)^{-1} : h^{-1} \to h^{-1}$ is a compact operator in h^{-1}. Thus $\hat{L}_v$ and therefore L_q have a compact resolvent. Hence, the spectrum of $\hat{L}_v$ is discrete, the corresponding eigenspaces are of finite dimension, and in every compact set $K \subset \mathbb{C}$ there are finitely many eigenvalues ([5], Chapter III, §6, 8). By Lemma 7, $\texttt{spec}(\hat{L}_v)$ is infinite and bounded from below. As v corresponds to a real valued potential q all the eigenvalues are real. In particular, $\texttt{spec}(\hat{L}_v) = \{\lambda_0(v) \le \lambda_1(v) \le ...\}$ and $\lambda_k(v) \to \infty$ as $k \to \infty$. This completes the proof of statements (i) and (ii).

In order to prove item (iii) we need Lemma 7. Taking a contour $\Gamma \subset Ext_M \cup \bigcup_{n>n_0} Vert_n(n)$ containing, for example, the first N eigenvalues of $\hat{L}_v$, $\lambda_0(v) \le \lambda_1(v) \le ... \le \lambda_N(v)$, consider the operators

$$Q^l_\Gamma(v) \stackrel{\text{def}}{=} \frac{1}{2\pi i} \oint_\Gamma \lambda^l(\lambda\mathbf{1} - \hat{L}_v)^{-1} d\lambda \in \mathcal{L}(h^{-1}),\ l = 0, 1, ..., N. \qquad (24)$$

The operators $Q^0_\Gamma(v), ..., Q^N_\Gamma(v) \in \mathcal{L}(h^{-1})$ are continuous as functions of $v \in h^{-1}$. The finite dimensional space $V(v) \overset{\text{def}}{=} \texttt{range}(Q^0_\Gamma(v))$ is invariant with respect to $Q^l_\Gamma(v)$ $(l = 0, 1, ..., N)$. In particular, the traces $\texttt{tr}(Q^l_\Gamma(v)|_{V(v)}) = \sum_{k=0}^N (\lambda_k(v))^l$ $(0 \le l \le N)$ and therefore, the eigenvalues $\lambda_0(v) \le \lambda_1(v) \le ... \le \lambda_N(v)$ are continuous which proves item (iii).

Using Lemma 7, we take a contour $\Gamma \subset Ext_M \cup \bigcup_{n>n_0} Vert_n(n)$ such that the simple eigenvalue $\lambda_k(v)$ is inside Γ and all other eigenvalues of $\hat{L}_v$ are outside of Γ.

Choose an eigenvector $w_k \in h^1$ of $\lambda_k(v)$ and a neighborhood $U(v)$ of v in h^{-1} so that for any $u \in U(v)$, $\lambda_k(u)$ is inside Γ, all other eigenvalues of $\hat{L}_u$ are outside Γ and $Q^0_\Gamma(u)w_k \ne 0$. Thus for any $u \in U(v)$, $Q^0_\Gamma(u)w_k \in h^1$ is an eigenvector of $\hat{L}_u$ corresponding to $\lambda_k(u)$. Clearly $u \mapsto Q^0_\Gamma(u)w_k \in h^1$ is analytic on $U(v)$ by Lemma 7 (as is the function $u \mapsto \lambda_k(u) = \texttt{tr}\, Q^1_\Gamma(u)$). By renormalizing $Q^0_\Gamma(u)w_k$ appropriately, item (iv) follows. $\square$

the finite dimensional on $V(u)$ has the $v_1, ..., v_d \in h^{-1}$ restriction $P(u)|_{V(u)}$ that depends analytically on u. $\lambda_k(u)$ and the a map $U(q) \to h^1$

Essentially the same arguments prove the following extension of Lemma 8 (ii).

Theorem 11. *The spectrum of Hill's operator* $L_q = -\frac{d^2}{dx^2} + q$ *on* $H^{-1}(\mathbb{T}^1_2, \mathbb{C})$ *with singular potential* $q \in H^{-1}(\mathbb{T}^1, \mathbb{C})$ *is discrete,* $\texttt{spec}(L_q) = \{\texttt{Re}(\lambda_0(q)) \le \texttt{Re}(\lambda_1(q)) \le \texttt{Re}(\lambda_2(q)) \le ...\}$*, the corresponding eigenspaces are of finite dimension, and* $\texttt{Re}(\lambda_k(q)) \to \infty$ *as* $k \to \infty$*.*

6.2 Dirichlet problem

The aim of the present section is to set up the Dirichlet problem for the operator $L_q \overset{\text{def}}{=} -\frac{d^2}{dx^2} + q$ on $[0, 1]$ with potential $q \in H^{-1}(\mathbb{T}^1)$. In order to make our presentation self-contained, we give some definitions and auxiliary facts on this problem in the present section. Further details (including the case $q \in H^{-\alpha}(\mathbb{T}^1)$, $\alpha < 1$) can be found in [3], §3.

Define the operator L^{Dir}_q on the Sobolev space $H^{-1}[0, 1]$ with domain $\texttt{Dom}(L^{Dir}_q) = H^1_{Dir}[0, 1]$. By definition, $H^1_{Dir}[0, 1] \overset{\text{def}}{=} \{f \in H^1[0, 1] \,|f(0) = f(1) = 0\}$ and $H^{-1}[0, 1] \overset{\text{def}}{=} (H^1_{Dir}[0, 1])'$ – see Section 1 where the definition of the Sobolev space $H^1[0, 1]$ is recalled.

In a natural way, the elements of $H^1_{Dir}[0,1]$ can be identified with elements in $H^1(\mathbb{T}^1)$, the corresponding inclusion map $\imath : H^1_{Dir}[0,1] \to H^1(\mathbb{T}^1)$ being continuous. For any $u \in H^1_{Dir}[0,1]$, qu is defined to be the functional in $H^{-1}[0,1] \overset{\text{def}}{=} (H^1_{Dir}[0,1])'$ given by the formula

$$\langle qu, v\rangle \overset{\text{def}}{=} \langle q, \imath(u)\imath(v)\rangle \,.$$

As the multiplication map $H^1(\mathbb{T}^1) \times H^1(\mathbb{T}^1) \to H^1(\mathbb{T}^1)$, $f \cdot g \overset{\text{def}}{=} fg$, is continuous, the mapping $H^{-1}[0,1] \times H^1_{Dir}[0,1] \to H^{-1}[0,1]$, $(q,u) \mapsto qu$, is continuous as is the operator $\frac{d^2}{dx^2} : H^1[0,1] \to H^{-1}[0,1]$. For any $u \in H^1_{Dir}[0,1]$ we set $L_q^{Dir} u \overset{\text{def}}{=} -\frac{d^2u}{dx^2} + qu$. In this way, L_q^{Dir} is a bounded operator $L_q^{Dir} : H^1_{Dir}[0,1] \to H^{-1}[0,1]$. Considered as an operator on $H^{-1}[0,1]$, L_q^{Dir} is an unbounded operator.

Lemma 9. *The operator L_q^{Dir} has compact resolvent. As a consequence, the spectrum of L_q^{Dir} is discrete, the eigenspaces are of finite dimension, and in every compact set $K \subset \mathbb{C}$ there are finitely many eigenvalues.*

Proof. As in §3.2 in [3] we identify the operator L_q^{Dir} with an operator on an appropriate sequence space. Then Lemma 9 can be proved using the same arguments as in the proof of Lemma 8. ☐

7 Appendix C: Convolution lemma

The convolution product $a * b$ of two sequences $a \overset{\text{def}}{=} \{a(k)\}_{k\in\mathbb{Z}}$ and $b \overset{\text{def}}{=} \{b(k)\}_{k\in\mathbb{Z}}$ is formally defined as the sequence given by $(a*b)(k) \overset{\text{def}}{=} \sum_{j\in\mathbb{Z}} a(k-j)b(j)$.

Lemma 10.

(i) For any $l \in \mathbb{Z}$ the convolution $(a,b) \mapsto a * b$ is continuous, uniformly in l, when viewed as a map $h^1 \times h^{-1,l} \to h^{-1,l}$ and $h^{-1} \times h^{1,l} \to h^{-1,l}$.

(ii) For any $\alpha \geq 0$, the convolution $(a,b) \mapsto a*b$ is continuous when viewed as a map $h^\alpha \times h^\alpha \to h^{\alpha-1}$.

(iii) For any $0 \leq \alpha \leq 1$, the convolution $(a,b) \mapsto a * b$ is continuous when viewed as a map $h^{\alpha+1} \times h^{\alpha-1} \to h^{\alpha-1}$.

The proof of Lemma 10 is straightforward.

$H^1(\mathbb{T}_2^1)$. and therefore,

$\lambda_0 \leq \lambda_1 \leq ...$ of L_q $\lambda_i = \lambda_j$,

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